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# Dependency on Rough Set Approximation Spaces

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ABSTRACT

In Rough Set Theory, dependency in the knowledge base is an important aspect when drawing inferences about the world. Rough set (inexact set) is defined by using a fixed binary relation (equivalence relation, tolerance relation, similarity relation, etc) to analyze vague, imprecise, inexact problems, which satisfy many interesting properties. The neighborhood based rough set theory is an extension of traditional rough set theory to handle with more complex practical problems. In this paper we discuss the dependency in the knowledge representation systems defined on three different I-neighborhood approximation operators and a covering approximation operator.

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## 1. INTRODUCTION

Rough set theory can be seen as a new mathematical approach to vagueness. Rough set philosophy is founded on the assumption that with every object in the universe of discourse we associate some information (data, knowledge). Objects characterized by same information are indiseernibility (similar) in view of the available information about them (Chellas [1]). The indiscernibility relation generated in this way is the mathematical basis of rough set theory.

The rough set theory has attracted many researchers all over the world, particularly for those of artificial intelligence. The rough set theory has been applied to many fields successfully such as Knowledge Discovery, Decision support, pattern recognition, Machine Learning Diagnosis, Biochemistry, Business Management, and Conflict Analysis etc.

Any set of all indiscernible (similar) objects is called an elementary set and forms a basic granule (atom) of knowledge about the universe (Tripathy et al [8]). Any union of some elementary sets is referred to as crisp (Precise) set, otherwise the set is rough (imprecise, vague).

The concept of rough set was introduced by Z. Pawlak ([6]) in 1982 to deal with imprecise, vague, uncertain problems. According to Prof. Pawlak, "Knowledge about a Universe can be considered as one's capability to classify objects of the Universe". By a classification of a Universe U, we mean a set of objects $\{X_1, X_2, ..............., X_n\}$ of $U$ such that $X_i \cap X_j = \phi$ for $i \neq j$ and $\bigcup_{i-1}^{n} X_i = U$. Let $R \subseteq U \times U$ denotes an equivalence relation on $U$, that is, $R$ is reflexive, symmetric and transitive relation. The equivalence class of an element $x \in U$ with respect to $R$, denoted by $[x]_R$, is the set of all $y \in U$ such that $xRy$. We denote U/R be the set of all equivalence classes of R. It two elements $x$, $y$ in $U$ belong to same equivalence class then we say that $x$ and $y$ are indistinguishable with respect to the relation $R$. Corresponding to every partition (or classification) of $U$ there is an equivalence relation, which has these partitions as its equivalence classes. As classification or partition of a Universe and the equivalence relations are interchangeable notions, the rough set is defined by equivalence relations as a mathematical point of view.

Let $U$ be finite, non-empty set of objects called the universe of discourse and R be an equivalence relation (knowledge or information) over $U$ called an indiscernibility relation. The set $U/R$, the family of all equivalence classes $R$, are referred to as categories or concepts of knowledge R and $[x]_R$ be a category in $R$, containing an element $x \in U$. An ordered pair $A = (U, R)$ is called an approximation space and a relational system $\kappa = (U, \mathfrak{R})$ is called a knowledge base, where U be the universe and $\mathfrak{R}$ be a family of equivalence relations defined on $U$. for $\mathrm{P} \subseteq \mathfrak{R} \cap \mathrm{P}$ (intersection of all equivalence relations belonging $P$) is an equivalence relation (indiscernibility relation) and is denoted by $IND(P)$. We have

$$[x]_{IND(\mathrm{P})} = \bigcap_{R \in \mathrm{P}} [x]_R .$$

## 2. ROUGH SET APPROXIMATIONS

Let $U$ be a non-empty finite set, called universe of discourse and R be an equivalence relation (Knowledge) on $U$. For any sub set $X$ of $U$, the lower approximation of $X$ in $A$ under the indiscernibility relation $R$ be defined by

(i) $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$

$= \bigcup\{Y \in U/R : Y \subseteq X\}$

and an upper approximation of $X$ in $A$ be defined by

(ii) $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$

$= \bigcup\{Y \in U/R : Y \cap X \neq \phi\}$

**Definition 2.1**: A set $X \subseteq U$ is called rough with respect to the knowledge $R$ (an equivalence relation $R$) if and only if $\underline{R}(X) \neq \overline{R}(X)$. The set $X$ is called definable with respect to R if and only if $\underline{R}(X) = \overline{R}(X)$ and $\overline{R}(X) - \underline{R}(X)$ in called the border line region of $X$ with respect to $R$ and is denoted by $BN_R(X)$.

Thus $\left(\underline{R}(X), \overline{R}(X)\right)$ is a rough set for $X$ under the available knowledge $R$ (an equivalence relation) for any subset $X$ of $U$.

The system $\left(2^U, \cap, \cup, -, \underline{R}, \overline{R}\right)$ is called rough set algebra, where $\cup, \cap, -$ are set intersection, union and complement respectively and $2^U$ be the set of all subsets of $U$.

The following properties of lower and upper approximations of rough set are well know (Pawlak [7]), for $X, Y \subseteq U$,

2(a) $\underline{R}\phi = \phi = \overline{R}\phi$, $\underline{R}U = U = \overline{R}U$

2(b) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$

2(c) $X \subseteq Y$ implies $\underline{R}(X) \subseteq \underline{R}(Y)$ and $\overline{R}(X) \subseteq \overline{R}(Y)$

2(d) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$ and $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$

2 (e) $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$ and $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$

2(f) $\overline{R}(-X) = -\underline{R}(X), \underline{R}(-X) = -\overline{R}(X)$

2(g) $\underline{R}\left(\underline{R}(X)\right) = \overline{R}\left(\underline{R}(X)\right) = \underline{R}(X), \overline{R}\left(\overline{R}(X)\right) = \underline{R}\left(\overline{R}(X)\right) = \overline{R}(X)$

The property (2(f)) shows that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Property (2(c)) states that the approximation operators are monotonic with respect to set inclusion.

## 3. DEPENDENCY

Let a system $\kappa = (U, \Re)$ be a knowledge base where $U$ is a universe of discourse and $\Re$ be a family equivalence relations defined on $U$ and let $P, \subseteq \Re$.

The knowledge $Q$ depends on the knowledge P, denotes $P \Rightarrow Q$, if and only if IND(P) $\subseteq$ IND(Q). That is any equivalence class of U/IND (P) is contained in one of the equivalence class of U/IND (Q). This is equivalent to say, the IND(P) boundary region of X is contained in the IND(Q) boundary region of X, for every subset X of U. That is $P \Rightarrow Q$ if and only if $BN_{IND(P)}(X) \subseteq BN_{IND(Q)}(X)$, $X \subseteq U$.

Knowledge $P$ and $Q$ are equivalent, denoted by $P \equiv Q$ if and only if $P \Rightarrow Q$ and $Q \Rightarrow P$ hold.

Obviously P $\equiv$ Q if and only if IND (P) = IND(Q)

Knowledge P and Q are independent, denoted by $P \neq Q$, if and only if neither $P \Rightarrow Q$ nor $Q \Rightarrow P$ hold.

**Example 1:** Let $U$ = {a,b,c,d,e,f,g} be the universe and the knowledge P,Q have the following partitions

U/P = {{a,c}, {b,g}, {d}, {e}, {f}} and

U/Q = {{a,c}, {b,f,q}, {d, e}}

Here P={$P$}, Q = {$Q$} $\subset$ R and $P$ = IND(P), $Q$ = IND(Q).

Hence IND(P) $\subseteq$ IND(Q), this is every partition under the knowledge P is contained in one of the partition under Q and then $P \Rightarrow Q$ or we write $P \Rightarrow Q$.

In this amazing world we do not get always the equivalence relation for the study of approximations. The neighborhood based rough set theory is an extension to the Pawlak rough net theory to analyze more practical problems. In the next section we present the notion of neighborhood approximations, defined by Lin[3], Yao[4], Chu[2], and others.

## 4. NEIGHBORHOOD APPROXIMATION OPERATOR–A

Let U be a universe of discourse, U be non empty and finite. A neighborhood operator n : U $\to$ $2^U$ assigns a unique neighborhood n(x) to each element x $\in$ U and n(x) is a nonempty subset of U which may or may not contain x. A neighborhood system of x, denoted by NS(x), is the maximal family of neighborhoods of x. If x has no neighborhood then NS(x) is an empty family, in this case we say that x has no neighborhood. A neighborhood system of U, denoted by NS(U), is the collection of NS(x) for all xÎU. The system (U, NS(U) is called neighborhood system space or simply neighborhood system.

In this article we write only 1- neighborhood system, that is, each element x $\in$ U has exactly one neighborhood in U. We find the following properties of a neighborhood operator n (Y.Yao[9]).

1. A neighborhood operator n is serial if for all $x \in U$, there exists a $y \in U$ such that $y \in n(x)$, that is, for all $x \in U$, $n(x) \neq \phi$.

2. The neighborhood operator n is inverse serial if for all $x \in U$, there exists a $y \in U$ such that $x \in n(y)$, $\bigcup_{x \in U} n(x) = U$.
3. The neighborhood operator n is reflexive if for all $x \in U$, $x \in n(x)$.
4. The neighborhood operator n is symmetric if for all $x, y \in U$, $x \in n(y) \Rightarrow y \in n(x)$.
5. The neighborhood operator n is transitive if for all $x, y, z \in U$

   $y \in n(x), z \in n(y), \Rightarrow z \in n(x)$
6. The neighborhood operator n is Euclidean if for all $x, y, z \in U$

   $y \in n(x), z \in n(x), \Rightarrow z \in n(y)$

A reflexive neighborhood operator is both serial and inverse serial. The family of neighborhoods $\{n(x) | x \in U\}$ of an 'inverse serial neighborhood operator' n forms a covering of the universe.

Let $n$ denotes an arbitrary 1-neighborhood operator and $n(x)$ be the corresponding neighborhood of $x \in U$. Then we define a pair of approximation operators (Lin and Yao[4], Mohanty [5]) for any subset $X$ of $U$,

(iii) $\underline{R}_n(X) = \{x \in U | n(x) \subseteq X\}$ =

$$\{x \in U | \forall y [y \in n(x) \Rightarrow y \in X]\}$$

(iv) $\overline{R}_n(X) = \{x \in U | n(x) \cap X \neq \phi\}$ =

$$\{x \in U | \forall y [y \in n(x) \Rightarrow y \in X]\}$$

For an equivalence relation R on U, the equivalence class $[x]_R$ may be considered as a neighborhood of $x \in U$. By substituting $[x]_R$ in stead of $n(x)$ we get the approximation operators (i) and (ii). Thus the approximation operators (iii) and (iv) are the generalization of (i) and (ii). The system $(2^U, \cap, \cup, -\underline{R}_n, \overline{R}_n)$ is called the rough set algebra. The subscript $n$ indicates that the approximation operators are defined and based on a particular neighborhood operator $n$.

Thus under the information available on U with respect to the neighborhood operator n, $(\underline{R}_n(X), \overline{R}_n(x))$ is a rough set for X, $X \subset U$. The border line region of X be given by

$$BN_n(x) = \overline{R}_n(x) - \underline{R}_n(x).$$

For an arbitrary neighborhood operator $n$, the pair of approximation operators satisfied the following properties (Yao[9]) ;

(a) $\underline{R}_n(X) = -(\overline{R}_n(-X))$

(b) $\overline{R}_n(X) = -(\underline{R}_n(-X))$

(c) $\underline{R}_n(U) = U, \overline{R}_n(\phi) = \phi$

(d) $\underline{R}_n(X \cap Y) = \underline{R}_n(X) \cap \underline{R}_n(Y)$

(e) $\overline{R}_n(X \cup Y) = \overline{R}_n(X) \cup \overline{R}_n(Y)$

(f) $\underline{R}_n(X \cup Y) \supseteq \underline{R}_n(X) \cup \underline{R}_n(Y)$

(g) $\overline{R}_n(X \cap Y) \subseteq \overline{R}_n(X) \cap \overline{R}_n(Y)$

(h) $X \subseteq Y$ implies $\underline{R}_n(X) \subseteq \underline{R}_n(Y)$ and

$\overline{R}_n(X) \subseteq \overline{R}_n(X)$

where X and Y are two subsets of U.

Additional properties of approximation operator $R_n$ are given below.

Suppose $n: U \to 2^U$ is I-neighborhood operator. If the neighborhood operator $n$ is serial then. $\overline{R}_n(\phi) = \phi$, $\overline{R}_n(U) = U$ and $\underline{R}_n(X) \subseteq \overline{R}_n(X)$.

The neighborhood operator n is inverse serial then for all $x \in U$, $R_n(-\{x\}) \neq U$ and $\overline{R}_n(\{x\}) \neq \phi$.

The neighborhood operator $n$ is reflexive then $\underline{R}_n(X) \subseteq \times \subseteq \overline{R}_n(X)$, $X \subseteq U$.

Now our aim is to define the dependency of knowledge by this neighborhood approximation operators.

Let $U$ be non-empty, finite universe, $n, k: U \to 2^u$ be two I-neighborhood operators. The union, denotes $n \bigcup k$, be a I-neighborhood operator, $n \bigcup k : U \to 2^U$ be defined by

$$(n \bigcup k)(x_j) = n(x_j) \bigcup k(x_j),$$

for each $x_j \in U$. Similarly a neighborhood operator, intersection, $n \bigcap k : U \to 2^U$ be defined by $(n \bigcap k)(x_j) = n(x_j) \bigcap k(x_j)$ for each $x_j \in U$.

**Definition 4.1:** Let $U$ be finite universe, $n, k : U \to 2^U$ be two neighborhood operators and $R_n$, $R_k$ be their corresponding approximation operators. The approximation operator $R_n$ depends upon the approximation operator $R_k$, denoted by $R_k \Rightarrow R_n$, if and only if $k(x_j) \subseteq n(x_j)$ for every element $x_j \in U$.

We note here that $k(x_j) \subseteq n(x_j)$ for each $x_j \in U$ if and only if $\underline{R}_k(X) \supseteq \underline{R}_n(X)$ For any set $X \subset U$ and $\overline{R}_k(X) \subseteq \overline{R}_n(X)$ for any $X \subset U$ This is equivalent to

$$BN_k(X) = \overline{R}_k(X) - \underline{R}_k(X) \subseteq \overline{R}_n(X) - \underline{R}_n(X) = BN_n(X)$$

for $X \subset U$.

**Definition 4.2 :** Let U be a finite universe, n, k : U → $2^U$ be two neighborhood operators. The approximation operators $R_n$ and $R_k$ are equivalent, denoted as $R_n \equiv R_k$ if and only if $R_n \Rightarrow R_k$ and $R_k \Rightarrow R_n$, also $R_n$ and $R_k$ are independent, denoted as $R_n \neq R_k$ if and only if neither $R_n \Rightarrow R_k$ nor $R_k \Rightarrow R_n$ hold.

**Proposition 4.1 :** Let U be a finite universe, n, k, p : U → $2^U$ be the neighborhood operators with serial and inverse serial property, and $R_n$, $R_k$, $R_p$ be their corresponding approximation operators Then.

(i) $R_k \Rightarrow R_n$ and $R_n \Rightarrow R_p$ implies $R_k \Rightarrow R_p$

(ii) $R_k \Rightarrow R_{k\cup n}$, $R_n \Rightarrow R_{k\cup n}$

(iii) $R_{k\cap n} \Rightarrow R_k$ and $R_{k\cap n} \Rightarrow R_n$, provided $k \cap n$ is an inverse serial operator.

(iv) $R_k \Rightarrow R_n$, $R_p \Rightarrow R_n$ implies $R_{k\cup p} \Rightarrow R_n$

(v) $R_k \Rightarrow R_n$, $R_k \Rightarrow R_p$ implies $R_k \Rightarrow R_{n\cap p}$, provided $n\cap p$ is an inverse serial operator.

(vi) $R_k \Rightarrow R_n$ if and only if $R_k \equiv R_{k\cup n}$.

**Proof :** For $R_k \Rightarrow R_n$ we get $k(x_j) \subseteq n(x_j)$ for all $x_j \in U$ and for $R_n \Rightarrow R_p$ we have $n(x_j) \subseteq p(x_j)$ for each $x_j \in U$.

Hence $k(x_j) \subseteq n(x_j) \subseteq p(x_j)$ for each $x_j \in U$, that is, $R_k \Rightarrow R_p$, $(i)$ is proved. Similarly other can be proved.

**Example 2:** Let $U = \{x, x_2, x_3, x_4, x_5, x_6\}$ be the universe. Let $k : U \to 2^U$ be a serial and inverse serial 1-neighborhood operator given by $k(x_1) = \{x_2, x_3\}, k(x_2) = \{x_2, x_4\}, k(x_3) = \{x_2\}$

$$k(x_4) = \{x_1, x_2\}, k(x_5) = \{x_3, x_5\}, k(x_6) = \{x_6\}.$$

Let n : U → $2^U$ be another serial and inverse serial 1-neighborhood operator such that

$$n(x_1) = \{x_2, x_3\},\ n(x_2) = \{x_2, x_4\},\ n(x_3) = \{x_2, x_4\},\ n(x_4) = \{x_1, x_2, x_5\},$$

$$n(x_5) = \{x_2, x_3, x_5\},\ n(x_6) = \{x_6\}.$$

Let $X = \{x_1, x_3, x_5\} \subset U$, then $\underline{R}_k(X) = \{x_5\}$, $\overline{R}_k(X) = \{x_4, x_5\}$ and $\underline{R}_n(X) = \phi$

$\overline{R}_n(X) = \{x_1, x_4, x_5\}$, so that

$$BN_k(X) = \overline{R}_k(X) - \underline{R}_k(X) = \{x_4\} \subset \{x_1, x_4, x_5\} = \overline{R}_n(X) - \underline{R}_n(X) = BN_n(X)$$

Hence $R_K \Rightarrow R_n$ *as* $k(x_j) \subseteq n(x_j)$ for each $x_j \in U$ and also $BN_k(X) \subseteq BN_n(X)$ holds.

Let $p : U \to 2^U$ be a serial and inverse serial I-neighborhood operator defined by,

$$p(x_1) = \{x_3\}, p(x_2) = \{x_2, x_4\}, p(x_3) = \{x_4\}, p(x_4) = \{x_1\}, p(x_5) = \{x_3, x_5\}, p(x_6) = \{x_6\},$$

Clearly $R_p \Rightarrow R_n$ *as* $P(x_j) \subseteq n(x_j)$ for each $x_j \in U$.

Now

$$k \cup p(x_1) = \{x_2, x_3\}, k \cup p(x_2) = \{x_2, x_4\}, k \cup p(x_3) = \{x_2, x_4\}, k \cup p(x_4) = \{x_1, x_2\}, k \cup p(x_5) = \{x_3, x_5\}, k \cup p(x_6) = (x_6)$$

Thus

$R_p \Rightarrow R_n$ and $R_k \Rightarrow R_n$ implies $R_{k\cup p} \Rightarrow R_n$,

as $k \cup p(x_j) \subseteq n(x_j)$ for each $x_j \in U$.

In the next section we will provide another 1-neighborhood system.

## 5. NEIGHBORHOOD APPROXIMATION OPERATOR-B

The neighborhood system in which each element of the universe has exactly one neighborhood, is termed as 1-neighborhood system.

New we define a 1-neighborhood system which is different from the previous section (Yao [9]) Let $\eta : U \to 2^U$ be a 1-neighborhood operator. For any $X \subseteq U$, the lower and upper approximation operators for the neighborhood operator $\eta$ be defined as

(v) $\eta\underline{R}(X) = \cup\{\eta(x) : x \in U, \eta(x) \subseteq X\}$

$$= \{x \in U : \exists y [x \in \eta(y), \eta(y) \subseteq X]\}$$

(vi) $\eta\overline{R}(X) = -\eta\underline{R}(-X)$

$$= \{x \in U : \forall y [x \in \eta(y) \Rightarrow \eta(y) \cap \times \neq \phi]\}$$

The lower and upper approximation operators $\eta\underline{R}, \eta\overline{R}$ satisfy the following properties, for $X, Y \subseteq U$,

(a) $\eta\underline{R}(X) = -\eta\overline{R}(-X), \eta\overline{R}(X) = -\eta\underline{R}(-X)$

(b) $\eta\underline{R}(X) \subseteq X, X \subseteq \eta\overline{R}(X)$

(c) $\eta\underline{R}(X) \subseteq \eta\underline{R}(\eta\underline{R}(X)), \eta\overline{R}(\eta\overline{R}(X)) \subseteq \eta\overline{R}(X)$

(d) $\eta\underline{R}(X \cap Y) \subseteq \eta\underline{R}(X) \cap \eta\underline{R}(Y)$

(e) $\eta\overline{R}(X \cup Y) \supseteq \eta\overline{R}(X) \cup \eta\overline{R}(Y)$

If $\eta$ is an inverse serial neighborhood operator, then $\eta\underline{R}(U) = U$, $\eta\overline{R}(\phi) = \phi$.

We have, under the information available with respect to the neighborhood operator $\eta,\left(\eta\underline{R}(x),\eta\overline{R}(x)\right)$ is a rough set for $X, X\subseteq U$ and $\left(2^U,\cap,\cup,-,\eta\underline{R},\eta\overline{R}\right)$ be the rough set algebra for the approximation operator $\eta R$.

We find a relation among these two neighborhood operators, defined in section – 4 and section – 5, be

$$\underline{R}_k(X)\subseteq k\underline{R}(X)\subseteq X\subseteq k\overline{R}(X)\subseteq \overline{R}_k(X),$$

whenever $k:U\to 2^U$ be a reflexive neighborhood operator.

**Example. 3:** Let $U$ = {1,2,3,4,5} be the universe and $k:U\to 2^U$ be the neighborhood operator, given by, k (1) = {1,2}, k(2) = {2,3}, k(3) = {3}, k (4) = {2,4}, k (5) = {1,3,5}. Clearly k is a reflexive operator on $U$.

For $X=\{1,2,5\},\underline{R}_k(X)=\{1\},\overline{R}_k(X)=\{1,2,4,5\}$ and

$k\underline{R}(X)=\{1,2\},k\overline{R}(X)=\{1,2,4,5\}$, so that,

$$\underline{R}_k(X)\subseteq k\underline{R}(X)\subseteq X\subseteq k\overline{R}(X)\subseteq \overline{R}_k(X)$$

Now we define dependency for the neighborhood approximation operator $\eta R$.

Let $\alpha,\beta:U\to 2^U$ be two neighborhood operators and $\alpha R,\beta R$ be their corresponding neighborhood operators, then $\beta R$ depends upon $\alpha R$, denotes $\alpha R\Rightarrow\beta R$ if an only if $\alpha(x_j)\subseteq\beta(x_j)$ for each $x_j\in U$ and $BN_{\alpha R}(X)\subseteq BN_{\alpha R}(X)\subseteq BN_{\beta R}(X)$ for all $X\subseteq U$.

The approximation operators $\alpha R$ and $\beta R$ are equivalent denoted by $\alpha R\equiv\beta R$, if and only if $\alpha R\Rightarrow\beta R$ and $\beta R\Rightarrow\alpha R$. Also $\alpha R$ and $\beta R$ are independent, denoted by $\alpha R\neq\beta R$ if and only if neither $\alpha R\Rightarrow\beta R$ non $\beta R\Rightarrow\alpha R$ hold.

**Proposition 5.1:** Let $U$ be a finite universe and $\alpha,\beta,\delta:U\to 2^U$ be the reflexive neighborhood operators and $\alpha R,\beta R,\delta R$ be their corresponding approximation operators. If $\alpha R\Rightarrow\beta R$ and $\beta R\Rightarrow\delta R$ then $\alpha R\Rightarrow\delta R$.

**Proof:** We have $\alpha R\Rightarrow\beta R$ and $\beta R\Rightarrow\delta R$. These implies $\alpha(x_j)\subseteq\beta(x_i)$ and

$\beta(x_j)\subseteq\delta(x_j)$ for each $x_j\in U$ and for any $X\subseteq U$,

$$BN_{\alpha R}(X)\subseteq BN_{\beta R}(X)$$

and $BN_{\beta R}(X)\subseteq BN_{\delta R}(X)$.

Thus

$BN_{\alpha R}(X)\subseteq BN_{\delta R}(X)$ and $\alpha(x_j)\subseteq\delta(x_j)$ for each $x_j\in U$. Hence $\alpha R\Rightarrow\delta R$.

**Example. 4:** Let $U$ = {a,b,c,d,e} be a universe and Let $\alpha,\beta:U\to 2^u$ be neighborhood operators given by

$\alpha(a)=\{a\},\alpha(b)=\{b\},\alpha(c)=\{c\},\alpha(d)=\{b,d\},\alpha(e)=$ {a, e} and $\beta(a)=\{a,b\},\beta(b)=\{b\}$,

$\beta(c)=\{c\},\beta(d)=\{b,d\},\beta(e)=\{a,b,c\}$.

Clearly $\alpha(x_j)\subseteq\beta(x_j)$

for each $x_j\in U$. Let $\alpha R$ and $\beta R$ be corresponding approximation operators for the neighborhood operators $\alpha,\beta$ respectively. Then for $X=\{a,d,e\}\subset U$, we get $\alpha\underline{R}(X)=\{a,e\},\alpha\overline{R}(X)=\{a,d,e\}$ and $\beta\underline{R}(X)=\phi,\beta\overline{R}(X)=\{a,d,e\}$. Thus for any $X\subseteq U$, we can find $BN_{\alpha R}(X)\subseteq BN_{\beta R}(X)$ that is, the borderline region of $X\subset U$ for the available knowledge $\alpha R$ is contained in the borderline region of $X$ for the knowledge $\beta R$ hence $\alpha R\Rightarrow\beta R$.

Lastly we write another 1-neighborhood system which is different from the neighborhood systems defined earlier. Also we deduce the dependencey for the new neighborhood approximation space in the following section.

### 6. NEIGHBORHOOD APPROXIMATION OPERATOR-C

Let $n:U\to 2^U$ be a 1-neighborhood operator. The lower and upper approximation operators, $\underline{apr}_n$ and $\overline{apr}_n$ respectively, for the neighborhood operator n be defined as, for any set $X\subseteq U$, (Y.Yao[9])

(Vii) $\overline{apr}_n(X)=\cup\{n(x)|x\in U,n(x)\cap X\neq\phi\}$

$$=\left\{x\in U\middle|\exists y\left[x\in n(y),n(y)\cap X\neq\phi\right]\right\}$$

(Viii) $\underline{apr}_n(X)=\ -\underline{apr}_n(-X)=$

$$\left\{x\in U\middle|\forall y\left[x\in n(y)\Rightarrow n(y)\subseteq X\right]\right\}$$

Then the set $X$ is called rough with respect to the approximation operator $apr_n$ if and only if $\underline{apr}_n(X)\neq\overline{apr}_n(X)$ and the set $X$ is exact (definable) if and only if $\underline{apr}_n(X)=\overline{apr}_n(X)$. The border line region of X, denoted by $BN_{An}(X)$, be $\overline{apr}_n(X)-\underline{apr}_n(X)$.

Thus $\left(\underline{apr}_n(X),\overline{apr}_n(X)\right)$ is a rough set for X and the system $\left(2^U,\cap,\cup,-,\underline{apr}_n,\overline{apr}_n\right)$ be the rough set algebra for the approximation operator $apr_n$. We find the following properties for the approximation operator $apr_n$, for $X,Y\subseteq U$.

(a) $\underline{apr}_n(X)=-\overline{apr}_n(-X)$, $\underline{apr}_n(X)=-\overline{apr}_n(-X)$

(b) $\underline{apr}_n(U)=U,\overline{apr}_n(\phi)=\phi$

(c) $\underline{apr}_n(X\cap Y)=\overline{apr}_n(X)\cap \underline{apr}_n(Y)$

(d) $\underline{apr}_n(X\cup Y)=\overline{apr}_n(X)\cup \overline{apr}_n(Y)$

(e) $\overline{apr}_n\left(\underline{apr}_n(X)\right)\subseteq X, X\subseteq \underline{apr}_n\left(\overline{apr}_n(X)\right)$

(f) $\underline{apr}_n\left(\underline{apr}_n(X)\right)\subseteq \underline{apr}_n(X), \overline{apr}_n(X)\subseteq$

$\overline{apr}_n\left(\overline{apr}_n(X)\right)$

If $n:U\rightarrow 2^u$ is an inverse serial neighborhood operator then $\underline{apr}_n(X)\subseteq X\subseteq \overline{apr}_n(X)$

**Example 5:** Suppose that a universe $U$ = {1,2,3}the neighborhood operator n be given by $n$ (1) = {1,2}, $n$(2) = {3}, $N$(3) = {2}.

Clearly $n$ is a serial and inverse serial neighborhood operator. For $X$ = {1,2}, we get

$\underline{apr}_n(X)=\{1,2\}, \overline{apr}_n(X)=\{1,2\}$ and for $Y=\{2,3\}$,

we have then $\underline{apr}_n(Y)=\{3\}, \overline{apr}_n(Y)=U$

Let $n:U\rightarrow 2^U$ be a 1-neighborhood operator and n be reflexive then we get a relation among there three neighborhood operators, for $X\subset U$,

$$\underline{apr}_n(X)\subseteq \underline{R}_n(X)\subseteq n\underline{R}(X)\subseteq X\subseteq n\overline{R}(X)\subseteq \overline{R}_n(X)\subseteq \overline{apr}_n(X).$$

Now we will define the dependency for the approximation operator $apr_n$ as follows.

Let $n,k:U\rightarrow 2^U$ two neighborhood operators and $apr_n$, $apr_k$ be their corresponding approximation operators. Then $\text{apr}_n$ depends on $apr_k$, denotes $apr_k\Rightarrow apr_n$, if and only if $k(x)\subseteq n(x)$ *for all* $x\in U$. consequently we get $BN_{Ak}(X)\subseteq BN_{An}(X)$ for each $X\subseteq U$.

The approximation operators $apr_k$ and $apr_n$ are equivalent, denotes $apr_k\equiv apr_n$, if and only if are independent, denoted by $apr_k\neq apr_n$, if and only if neither $apr_k\Rightarrow apr_n$ *non* $apr_n\Rightarrow apr_k$ hold.

**Proposition 6.1:** Let $U$ be a finite universe, $n,k,p:U\rightarrow 2^U$ be 1-neighborhood operators with serial and inverse serial property and $apr_n$, $apr_k$, $apr_p$ be their corresponding approximation operators, then

(i) $apr_k\Rightarrow apr_n$ and $apr_n\Rightarrow apr_p$ implies $apr_k\Rightarrow apr_p$

(ii) $apr_k\Rightarrow apr_{k\cup n}, apr_n\Rightarrow apr_{k\cup n}$.

(iii) $apr_k\Rightarrow apr_n, apr_p\Rightarrow apr_n$ *implies* $apr_{k\cup p}\Rightarrow apr_n$

(iv) $apr_k\Rightarrow apr_n$ if and only if $apr_n\Rightarrow apr_{k\cup n}$.

(v) $apr_{k\cap n}\Rightarrow apr_k$, provided $k\cap n$ is an inverse serial operator.

(vi) $apr_k\Rightarrow apr_n, apr_k\Rightarrow apr_p$ *implies* $apr_k\Rightarrow apr_{n\cap p}$ provided $n\cap p$ is an inverse serial operator.

**Proof :** We have $apr_k\Rightarrow apr_n$ *and* $apr_p\Rightarrow apr_n$.

So that $k(x)\subseteq n(x)$ *and* $p(x)\subseteq n(x)$ *for all* $x\in U$. This implies $k(x)\cup p(x)\subseteq n(x)$ that is $(k\cup p)(x)\subseteq n(x)$ and hence $apr_{k\cup p}\Rightarrow apr_n$ this completes the proof of (iii).

Similarly other can be proved.

**Example–6:** Let $U$ = {$a,b,c,d,e$} be the universe and $n,k:U\rightarrow 2^U$ be two 1-neighborhood operators, given by, $n(a)$ = {$a$, $b$}, $n(b)$ = {$c$, $d$} $n(c)$ = {$c$}, $n(d)$ = {$a,c$}, $n(e)$ = {$a,d,e$} and $k(a)$ = {$a,b$}, $k(b)$ = {$d$}, $k${$c$} = {$a$, $d$}, $k(d)$ = {$a,b,c$}, $k(e)$ = {$b,c,d,e$}.

Then $(k\cup n)(a)$ = {$a,b$}, $(k\cup n)$ $(b)$ = {$c,d$}, $(k\cup n)$ $(c)$ = {$a,c,d$}, $(k\cup n)$ $(d)$ = {$a,b,c$}, $(kun)$ $(e)$ = {$a,b,c,d,e$) clearly

$apr_k\Rightarrow apr_{k\cup n}$ and $apr_n\Rightarrow apr_{k\cup n}$, as $k(x)\subseteq(k\cup n)(x)$ and $n(x)\subseteq(k\cup n)(x)$ for eac$h$ $x\in U$.

## 7. COVERING BASED ROUGH SET

We require the following definition to define the covering based rough sets(W.Zhu et.al.[10] )

**Definition 7.1 :** Let $U$ be a universe of discourse and $C$ be a family of nonempty subset of $U$. $C$ is called a cover of $U$ if $\cup C$ = $U$.

We call ($U$,$C$) be the covering approximation space and the covering $C$ is called the family of approximation sets.

It is clear that a partition of $U$ is certainly a covering of $U$, so the concept of a covering is an extension of a partition.

**Definition 7.2 :** Let ($U$, $C$) be an approximation space and $x$ be any element of $U$. then the family $Md(x)=$ $\{K\in C: x\in K\Lambda\forall S\in C[x\in S\Lambda S\subseteq K\Rightarrow K=S]\}$ is called the minimal description of the object $x$.

In order to describe an object we need only the essential characteristics related to this object. This is the purpose of the minimal description concept.

**Example 7:** Let ($U$, $C$) be an approximation space, where $U$ = {$a$, $b$, $x$, $y$, $z$} and

$C$ = {{$a$, $b$, $x$}, {$b$, $x$}, {$x$, $y$}, {$b$, $z$}, {$a$, $y$, $z$}}

Clearly $\cup C$ = $U$ then for $x\in U$

$Md(x) = \{\{b, x\}, (x, y\}\}$, since $\{a, b, x\} \supset \{b, x\}$

be the minimal description of the element $x$.

**Definition 7.3:** For any set $X \subseteq U$, the family of sets $C_*(X) = \{K \in C : K \subseteq X\}$ is called bottom approximation of the set $X$.

**Definition 7.4:** For all $X \subseteq U$**,** the set $X_* = \cup C_*(X)$ is called lower approximation of the set $X$.

**Definition 7.5:** Let $(U, C)$ be a covering approximation space. For approximation set $X \subseteq U$, the covering upper approximation of $X$ be defined by $X^* = \cup\{Md(x) : x \in X\}$.

The set $X$ is called covering based rough if $X_* \neq X^*$, otherwise $X$ is called exact set with respect to covering $C$.

The boundary of X be given by $BN_C(X) = X^* - X_*$ is known as the border line region of $X$ for the covering.

We find the following properties for the covering lower and upper approximations.

**Proposition 7.1 :** For $X, Y \subseteq U$,

(a) $U_* = U, U^* = U$

(b) $Q_* = Q,\ Q^* = Q$

(c) $X_* =\subseteq X,\ \subseteq X \subseteq X^*$

(d) $(X_*)_* = X_*,\ (X^*)^* = X^*$

(e) $X \subseteq Y \Rightarrow X_* \subseteq Y_*$ *and* $X^* \subseteq Y^*$

**Example 8:** Let $U = (a, b, c, d\}$ and $C = \{(a, b), (b, c), (c), \{d\}\}$ be a cover of $U$.

Let $X = \{a, c\}$, then $X_* = \phi$ and $X^* = \{a,b,c\}$

$Y = \{a,b\}$, then $Y_* = \{a,b\}$, $Y^* = \{a,b,c\}$

Let $C_1 = \{\{a, b\}, \{b, c, d\}, \{d\}\}$ be another corer of $U$, then for $Y = \{a,b\}$, $Y_* = \{a,b\}$, $Y^* = \{a,b,c,d\}$ be the lower and upper approximation of $Y$ under cover $C_1$.

For $Z = \{c\}$, $C - Z_* = \{c\}$, $C - Z^* = \{c\}$ be the lower and upper approximation of $Z$ under the covering $C$ and $C_1 - Z_* = \phi$, $C_1 - Z^* = \{a,c,d\}$ under the covering $C_1$.

Let $U$ be the nonempty finite universe and Let $C_1$, $C_2$ be two covering of $U$. we write the covering $C_1$ is contained in the covering $C_2$, denotes $C_1 \subseteq C_2$, if and only if for every $K \in C_1$ there exists at least one $T \in C_2$ such that $K \subseteq T$.

**Definition 7.6:** Let $C_1$, $C_2$ be two coverings for $U$. There the covering $C_1$ depends on $C_2$, denoted by $C_2 \Rightarrow C_1$ if and only if $C_2 \subseteq C_1$ and $BN_{C_2}(X) \subseteq BN_{C_1}(X)$ for all $X \subseteq U$.

**Proposition 7.2:** Let $U$ be a universe and let $(U, C_1)$, $(U, C_2)$, $(U, C_3)$ be covering approximation spaces, where $C_1$, $C_2$, $C_3$ be three different coverings for $U$. Then

(i) $C_2 \Rightarrow C_1$ *and* $C_3 \Rightarrow C_2$ *implies* $C_3 \Rightarrow C_1$

(ii) $C_2 \Rightarrow C_1$ *and* $C_3 \subseteq C_1$ *implies* $C_2 \Rightarrow C_3$

(iii) $C_2 \Rightarrow C_1$ *and* $C_2 \subseteq C_3$ *implies* $C_3 \Rightarrow C_1$

we note here that $C_2 \subseteq C_1$ implies $BN_{C_2}(X) \subseteq BN_{C_1}(X)$ for any $X \subseteq U$ when $C_2 \subseteq C_1$ we get $C_2 - X_* \supseteq C_1 - X_*$ for any $X \subseteq U$ and also $C_2 - X^* \subseteq C_1 - X^*$ so that $BN_{C_2}(X) \subseteq BN_{C_1}(X)$.

**Proof:** For $C_2 \Rightarrow C_1$ and $C_3 \Rightarrow C_2$ we get $C_2 \subseteq C_1$, $C_3 \subseteq C_2$ and $BN_{C_3}(X) \subseteq BN_{C_1}(X)$ for any $BN_{C_3}(X) \subseteq BN_{C_2}(X)$ for any $X \subseteq U$. So that $C_3 \subseteq C_1$ and $BN_{C_3}(X) \subseteq BN_{C_1}(X)$ for any $X \subseteq U$ and here $C_3 \Rightarrow C_1$. (i) is proved.

## 7. CONCLUSION

Dependencies in a knowledge base are basic tools when drawing conclusions from basic knowledge, for they state some of the relationships between basic categories in the knowledge base. Inference rules is of the form if ………then……….., this aspect can be formulated as how, from a given knowledge, another knowledge can be induced. In this paper we discuss the dependencies in three different 1-neighborhood approximation spaces. In particular. $\eta R$ -approximation space (section – 5) be more appropriate one in which the borderline region for any $X \subseteq U$ (universe) be thinner than the other two neighborhood systems. We find several examples that a given knowledge can be induced in knowledge.

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