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FUZZY DOUBLE DOMINATION NUMBER AND CHROMATIC NUMBER OF A FUZZY GRAPH

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A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to at least one vertex in S. A Dominating set is said to be Fuzzy Double Dominating set if every vertex in V-S is adjacent to at least two vertices in S. The minimum cardinality taken over all, the minimal double dominating set is called Fuzzy Double Domination Number and is denoted by $\gamma_{fdd}(G)$. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\chi(G)$. For any graph G a complete subgraph of G is called a clique of G. In this paper we find an upper bound for the sum of the Fuzzy Double Domination Number and Chromatic Number in fuzzy graphs and characterize the corresponding extremal fuzzy graphs.

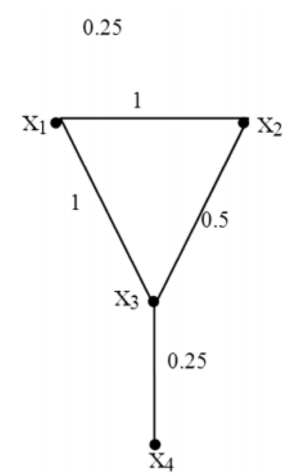
Keywords: Fuzzy Double Domination Number, Chromatic Number, Clique, Fuzzy Graphs.

1. Introduction

Let G = (μ, σ) be a simple undirected fuzzy graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by d(u). The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively, P_n denotes the path on n vertices. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. The Chromatic Number χ is defined to be the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour. For any graph G a complete subgraph of G is called a clique of G. The number of vertices in a largest clique of G is called the clique number of G.

A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to atleast one vertex in S. The minimum cardinality taken over all dominating sets in G is called the domination number of G and is denoted by γ. A Dominating set is said to be Fuzzy Double Dominating set if every vertex in V-S is adjacent to atleast two vertices in S. The minimum cardinality taken over all, the minimal double dominating set is called Fuzzy Double Domination Number and is denoted by γ_{fdd} (G).

If X is a collection of objects denoted generically by x, then a Fuzzy set $\bar{A}$ in X is a set of ordered pairs: $\bar{A} = \{(x, \mu_{\bar{A}}(x))/x \in X\}$, $\mu_{\bar{A}}(x)$ is called the membership function of x in $\bar{A}$ that maps X to the membership space M (when M contains only the two points 0 and 1). Let E be the (crisp) set of nodes. A Fuzzy graph is then defined by, $\bar{G}(x_i, x_j) = \{((x_i, x_j), \mu_{\bar{G}}(x_i, x_j))/(x_i, x_j) \in E \times E\}$. $H(x_i, x_j)$ is a Fuzzy Subgraph of $\bar{G}(x_i, x_j)$ if $\mu_{\bar{H}}(x_i, x_j) \leq \mu_{\bar{G}}(x_i, x_j)$ $\forall (x_i, x_j) \in E \times E$, $\bar{H}(x_i, x_j)$ spans graph $\bar{G}(x_i, x_j)$ if the node set of $\bar{H}(x_i, x_j)$ and $\bar{G}(x_i, x_j)$ are equal, that is if they differ only in their arc weights.



$\mu(x_1) = 0.1, \mu(x_2) = 0.5, \mu(x_3) = 0.4, \mu(x_4) = 0.2$

Fuzzy Graph $\bar{G}$

The first definition of Fuzzy graphs was proposed by Kaufmann[4], from the fuzzy relations introduced by Zadeh[9]. Although Rosenfeld[5] introduced another elaborated definition, including fuzzy vertex and fuzzy edges. Several fuzzy analogs of graph theoretic concepts such as paths, cycles connectedness etc. The concept of domination in fuzzy graphs was investigated by A.Somasundaram, S.Somasundaram [6]. A. Somasundaram present the concepts of independent domination, total

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domination, connected domination and domination in Cartesian product and composition of fuzzy graphs([7],[8]).

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [10], Paulraj Joseph J and Arumugam S proved that $\gamma + k \leq p$. In [9], Paulraj Joseph J and Arumugam S proved that $\gamma_c + \chi \leq p + 1$. They also characterized the class of graphs for which the upper bound is attained. They also proved similar results for γ and γ_t. In [13], Mahadevan G introduced the concept of complementary perfect domination number γ_{cp} and proved that $\gamma_{cp} + \chi \leq 2n - 2$, and characterized the corresponding extremal graphs. In [12], Mahadevan G, Selvam A, Mydeen bibi A, proved that $\gamma_{dd} + \chi \leq 2n$. They also characterized the class of graphs for which the upper bound is attained. In this paper, we obtain sharp upper bound for the sum of the Fuzzy Double Domination Number and chromatic number and characterize the corresponding extremal Fuzzy graphs. We use the following previous results.

Theorem 1.1 [1]: For any connected graph G, γ_{dd} (G) $\leq$ n.

Theorem 1.2 [2]: For any connected graph G, $\chi(G) \leq \Delta(G) + 1$.

2. MAIN RESULTS

Theorem 2.1: For any connected fuzzy graph G, γ_{fdd} (G) + χ(G) $\leq$ 2n and the equality holds if and only if $G \cong K_2$.

Proof: $\gamma_{fdd}(G) + \chi(G) \leq n + \Delta + 1 = n + (n-1) + 1 \leq 2n$. If $\gamma_{fdd}(G) + \chi(G) = 2n$. Then the only possible case is $\gamma_{fdd} = n$ and $\chi = n$. Since $\chi = n$, $G = K_n$, But for K_n, $\gamma_{fdd} = 2$, so that $G \cong K_2$. Converse is obvious.

Theorem 2.2: For any connected fuzzy graph G, $\gamma_{fdd}(G) + \chi(G) = 2n - 1$ if and only if $G \cong K_3$.

Proof: Assume that $\gamma_{fdd}(G) + \chi(G) = 2n - 1$. This is possible only if $\gamma_{fdd} = n$ and $\chi = n - 1$ (or) $\gamma_{fdd} = n - 1$, $\chi = n$.

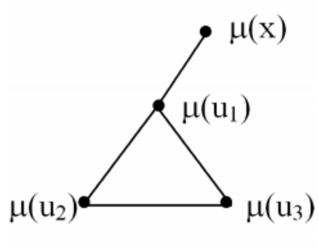
Case (i) Let $\gamma_{fdd} = n$ and $\chi = n - 1$.

Since $\chi = n - 1$, G contains a clique K on n – 1 vertices. Let x be a vertex other than the vertices of K_{n-1}. Since G is connected, x is adjacent to u_i for some i in K_{n-1}. Then $\{x, u_i, u_j\}$ is a γ_{fdd} – set, so that $\gamma_{fdd} = 3$. Since $\gamma_{fdd} = n$, we have n = 3. Hence $K = K_2$. Let u, v be the vertices of K_2. Let x be adjacent to u. Then $\gamma_{fdd} = 2$, which is a contradiction. Hence no fuzzy graph exists.

Case (ii) If $\gamma_{fdd} = n - 1$ and $\gamma = n$.

Since $\chi = n$, $G = K_n$. But for K_n, $\gamma_{fdd}(G) = 2$, so that n = 3. Hence $G \cong K_3$. Converse is obvious.

Theorem 2.3: For any connected fuzzy graph G, $\gamma_{fdd}(G) + \chi(G) = 2n - 2$ if and only if K_4 or G_1 given in figure 2.1.



G_1

Fig. 2.1

Proof: If G is either K_4 or G_1, then clearly $\gamma_{fdd}(G) + \chi(G) = 2n - 2$. Conversely, assume that $\gamma_{fdd}(G) + \chi(G) = 2n - 2$. This is possible only if $\gamma_{fdd} = n$ and $\chi = n - 2$ (or) $\chi_{fdd} = n - 1$ and $\chi = n - 1$ (or) $\gamma_{fdd} = n - 2$ and $\chi = n$.

Case (i) Let $\gamma_{fdd} = n$ and $\chi = n - 2$.

Since $\chi = n - 2$, G contains a clique K on n – 2 vertices. Let $S = \{x, y\} \in V\text{-}S$. Then $<S> = K_2$ or $\overline{K}_2$.

Subcase (a) Let $<S> = K_2$. Since G is connected, x is adjacent to some u_i of K_{n-2}. Then $\{y, u_i, u_j\}$ for $i \neq j$ in K_{n-2} is a γ_{fdd} -set, so that $\gamma_{fdd} = 3$ and hence n = 3. But $\chi = n - 2 = 1$, which is a contradiction. Hence no fuzzy graph exists.

Subcase (b) Let $<S> = \overline{K}_2$. Since G is connected, x is adjacent to some u_i of K_{n-2}. Then y is adjacent to the same u_i of K_{n-2} or adjacent to u_j of K_{n-2} for $i \neq j$. In both the cases $\{x, y, u_i, u_j\}$ is a γ_{fdd} set. Since $\gamma_{fdd} = n$, we have n = 4. Hence $K = K_2$. Let u, v be the vertices of K_2. Without loss of generality, let x and y both be adjacent to u. Then $\gamma_{fdd} = 3$, which is a contradiction. Hence no fuzzy graph exists. Now without loss of generality, let x be adjacent to u and y be adjacent to v. In this case also no fuzzy graph exists.

Case (ii) Let $\gamma_{fdd} = n - 1$ and $\chi = n - 1$.

Since $\chi = n - 1$, G contains a clique K on n – 1 vertices. Let x be a vertex other than the vertices of K_{n-1}. Since G is connected, x is adjacent to u_i for some i in K_{n-1}. Then $\{x, u_i, u_j\}$ is a γ_{fdd} -set, so that $\gamma_{fdd} = 3$. Hence n = 4, so that $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Then x is adjacent to one of u_i, say u_1. If d(x) = 1 then $G \cong G_1$. If x is adjacent to one more vertex say u_2, then $\{x, u_3\}$ is a γ_{dd} -set, which is a contradiction.

Case (iii) Let $\gamma_{fdd} = n - 2$ and $\chi = n$.

Since $\chi = n$, $G = K_n$. But for K_n, $\gamma_{fdd}(G) = 2$, so that n = 4. Hence $G \cong K_4$.

Theorem 2.4: For any connected graph G, $\gamma_{fdd}(G) + \chi(G) = 2n - 3$ if and only if $G \cong P_4$ or any one of the following fuzzy graphs in the figure 2.2.

Proof : If G is any one of the graph given in the figure, then clearly $\gamma_{fdd}(G) + \chi(G) = 2n - 3$. Conversely assume that

γ_{fdd} (G) + χ(G) = 2n – 3. This is possible only if γ_{fdd} = n and χ = n – 3 (or) γ_{fdd} = n – 1 an χ = n – 2 (or) γ_{fdd} = n – 2 and χ = n – 1 (or) γ_{fdd} = n – 3, χ = n.

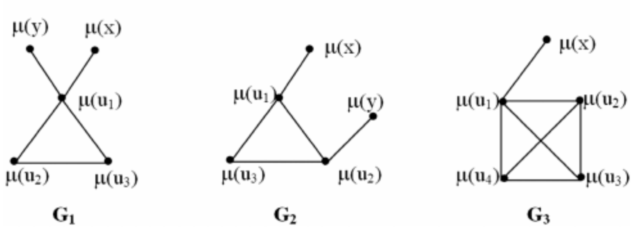


Fig. 2.2

Case (i) Let γ_{fdd} = n and χ = n – 3.

Since χ = n – 3, G contains a clique K on n – 3 vertices. Let S = {x, y, z}$\in$V - S.

Then <S> = K_3, $\overline{K}_3$, P_3, $K_2 \cup K_1$

Subcase (i) Let <S> = K_3. Since G is connected, x is adjacent to some u_i in K_{n-3}. Then {x, y, u_i, u_j} for i $\neq$ j in K_{n-3} is a γ_{fdd} - set of G, so that γ_{fdd} = 4. Hence n = 4. In that case χ = 1, so that G is totally disconnected, which is a contradiction. Hence no fuzzy graph exists.

Subcase (ii) Let <S> = $\overline{K}_3$. Since G is connected, one of the vertices of K_{n-3} say u_i, is adjacent to all the vertices of S or two vertices of S or one vertex of S. If u_i for some i is adjacent to all the vertices of S, then {x, y, z, u_i, u_j} for i $\neq$ j in K_{n-3} is a γ_{fdd} -set of G. If u_i is adjacent to two vertices of S say x and y then since G is connected, z is adjacent to u_j for i $\neq$ j in K_{n-3}, then {x, y, z, u_i, u_j} for i $\neq$ j in K_{n-3} is a γ_{fdd} set of G. If u_i is adjacent to x and u_j is adjacent to y and u_k is adjacent to z then {x, y, z, u_i, u_j} for i $\neq$ j $\neq$ k in K_{n-3} is a γ_{fdd} set of G. In all the cases n = 5, so that γ_{fdd} = 4, which is a contradiction. Hence no fuzzy graph exists.

Subcase (iii) Let <S> = P_3 = (x, y, z). Since G is connected, x(or equivalently z) is adjacent to u_i for some i in K_{n-3}. Then {x, z, u_i, u_j} for i $\neq$ j in K_{n-3} is a γ_{fdd} -set of G. If u_i is adjacent to y then {x, y, u_i, u_j} for i $\neq$ j is a γ_{fdd} - set. In all the cases, n = 4, so that χ = 1, for which G is totally disconnected, which is a contradiction. Hence no graph exists.

Subcase (iv) Let <S> = $K_2 \cup K_1$. Let xy be the edge in $K_2 \cup K_1$, since G is connected. There exists a u_i in K_{n-3} is adjacent to x. If z is adjacent to same u_i, then {y, z, u_i, u_j} for i $\neq$ j is a γ_{fdd} - set. If z is adjacent to u_j for some i $\neq$ j then {y, z, u_i, u_j} for i $\neq$ j is a γ_{fdd} set. In all the cases, n = 4, so that χ = 1 for which G is totally disconnected. Hence no graph exists.

Case (v) Let γ_{fdd} = n – 1 and χ = n – 2.

Since χ = n – 2, G contains a clique K on n – 2 vertices. Let S = {x, y} $\in$ V-S. Then <S> = K_2 or $\overline{K}_2$.

Subcase (a) Let <S> = K_2. Since G is connected, x (or equivalently y) is adjacent to some vertex u_i of K_{n-2}. Then {y, u_i, u_j} for i $\neq$ j in K_{n-2} is a γ_{fdd} -set, so that n = 4. Hence K = K_2 = uv. If x is adjacent to u, then G $\cong$ P_4.

Subcase (b) Let <S> = $\overline{K}_2$. Since G is connected, x is adjacent to some u_i of K_{n-2}. If y is adjacent to the same u_i of K_{n-2} or adjacent to u_j for i $\neq$ j then in both the cases {x, y, u_i, u_j} for i $\neq$ j is a γ_{fdd} - set, so that γ_{fdd} = 4. Hence n = 5. Hence K = K_3. Let u_1, u_2, u_3 be the vertices of K_3. If x and y are adjacent to a common vertex say u_1. If d(x) = d(y) = 1, then G $\cong$ G_1. If x is adjacent to u_1 and y is adjacent to u_2 and if d(x) = d(y) = 1, then G $\cong$ G_2. For all other cases are not possible.

Case (iii) Let γ_{fdd} = n – 2 and χ = n – 1.

Since χ = n – 1, G contains a clique K on n – 1 vertices. Let x be a vertex not in K_{n-1}. Since G is connected, x is adjacent to u_i for some i in K_{n-1}. Then {x, u_i, u_j} is a γ_{fdd} -set, so that n = 5. Hence K = K_4. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Then x must be adjacent to exactly one vertex say u_1 of K_4. Hence G $\cong$ G_3.

Case (iv) Let γ_{fdd} = n – 3 and χ = n.

Since χ = n, G = K_n. But for K_n, γ_{fdd} = 2 so that n = 5. Hence G $\cong$ K_5.

Theorem 2. 5: For any connected graph G, γ_{fdd} (G) + χ(G) = 2n – 4 if and only if G $\cong$ K_6 or any one of the following graphs given in the figure 2.3.

Proof: If G is any one of the graph given in the figure, then clearly γ_{fdd} (G) + χ(G) = 2n – 4. Conversely assume that γ_{fdd} (G) + χ(G) = 2n – 4. This is possible only if, γ_{fdd} = n and χ = n – 4 (or) γ_{fdd} = n – 1 and χ = n – 3 (or) γ_{fdd} = n – 2 and χ = n – 2 (or) γ_{fdd} = n – 3 and χ = n – 1 (or) γ_{fdd} = n – 4 and χ = n.

Case (i) If χ_{fdd} = n and χ = n – 4.

Since χ = n – 4, G contains a clique K on n – 4 vertices. S = {v_1, v_2, v_3 , v_4}$\in$V-S. Then <S> = K_4, $\overline{K}_4$, $K_3 \cup K_1$, P_4, $K_{1,3}$, $P_3 \cup K_1$, $K_2 \cup K_2$.

In all the above cases, it can be verified that no new graph exits.

Case (ii) Let γ_{fdd} = n –1 and χ = n – 3.

Since χ = n – 3, G contains a clique K on n – 3 vertices. Let S ={x, y, z}$\in$ V-S. Then <S> = K_3, $\overline{K}_3$, P_3, $K_2 \cup K_1$.

If <S> = K_3, then no graph exists.

Subcase (a) Let <S> = $\overline{K}_3$. Since G is connected, one of the vertices of K_{n-3} say u_i, is adjacent to all the vertices of S(or) two vertices of S(or) one vertex of S. In all the cases, {x, y, z, u_i, u_j}for i $\neq$ j is a γ_{fdd} - set of G, so that γ_{fdd} = 5. Hence n = 6, so that K = K_3. Let u_1, u_2, u_3 be the vertices of K_3. Without loss of generality, let u_1 be adjacent to all the

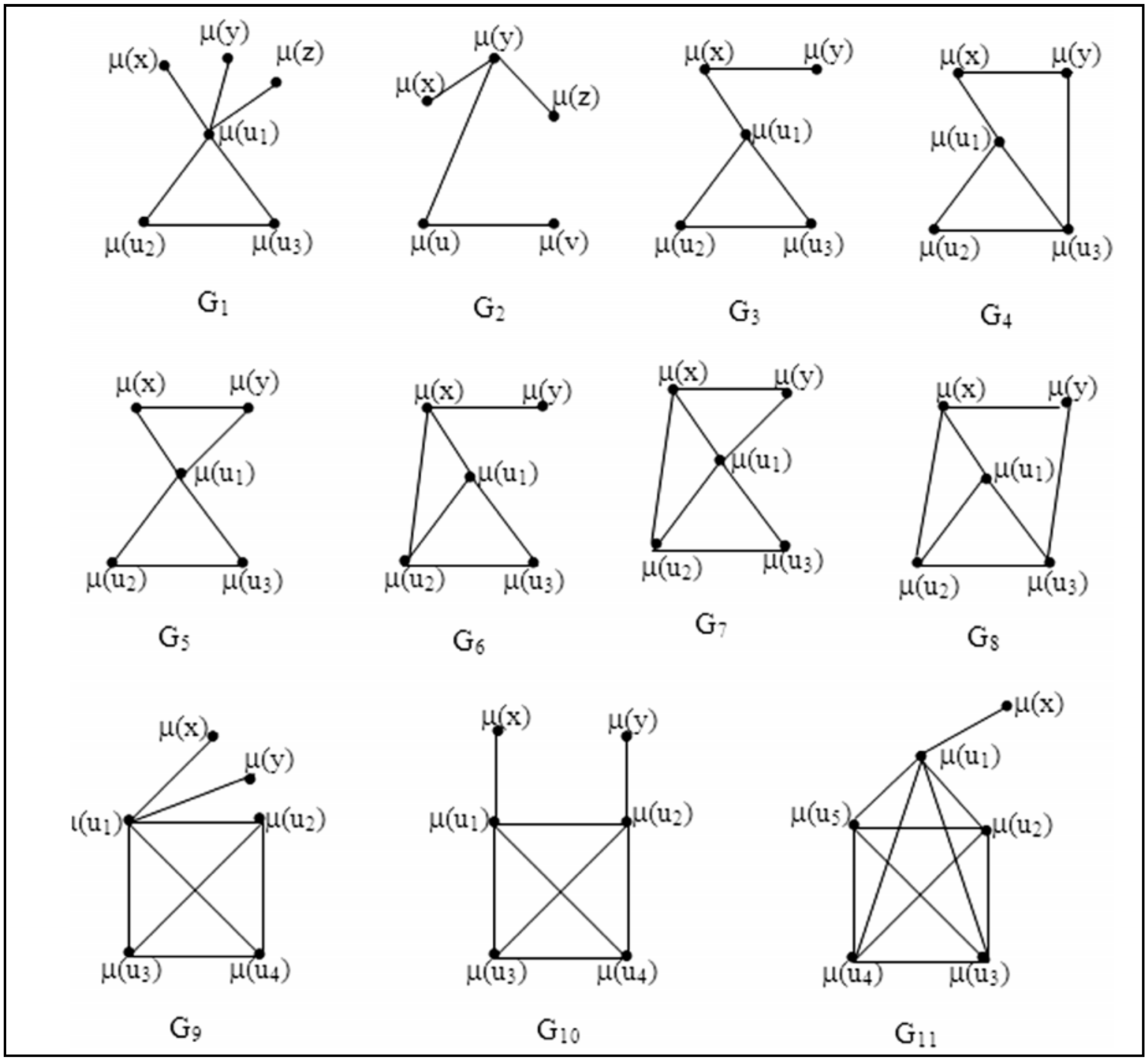


Fig.2.3

vertices of S and if d(x) = d(y) = d(z) = 1, then $G \cong G_1$. In all other cases, no new graph exits.

Subcase (b) Let <S> = P_3 = (x, y, z). Since G is connected, x (or equivalently z) is adjacent to u_i for some i in K_{n-3}. Then {x, z, u_i, u_j} for $i \neq j$ is a γ_{fdd} -set, so that n = 5. In this case no graph exists. If u_i is adjacent to y in S then {x, z, u_i, u_j} for $i \neq j$ is a γ_{fdd} -set, so that n = 5. Hence K = K_2 = uv. If u is adjacent to y, then $G \cong G_2$ and in all other cases no new graph exits.

Subcase (c) Let <S> = $K_2 \cup K_1$. Let xy be the edge in $K_2 \cup K_1$. Since G is connected, there exists an u_i in K_{n-3} is adjacent to x. if z is adjacent to same u_i or z is adjacent to u_j for $i \neq j$ in K_{n-3}. Then {y, z, u_i, u_j} for $i \neq j$ is a γ_{fdd} - set. n = 5. K = K_2 = uv. Let x be adjacent to u. Since G is connected, z is adjacent to u or v. If z is adjacent to u, then $G \cong G_2$. If z is adjacent to v, then $\gamma_{fdd} = 3$, which is a contradiction. For all the remaining cases, no new graph exits.

Case (iii) Let $\gamma_{fdd} = n - 2$ and $\chi = n - 2$.

Since $\chi = n - 2$, G contains a clique K on n – 2 vertices. Let S = {x, y}$\in$V-S. Then <S> = K_2 or $\overline{K}_2$.

Subcase (a) Let <S> = K_2, Since G is connected, x is adjacent to some u_i in K_{n-2}. Then {y, u_i, u_j} for $i \neq j$ in K_{n-2} is a γ_{fdd} -set, so that n = 5. Hence K = K_3. Let u_1, u_2, u_3 be the vertices of K_3. Let x be adjacent to u_1, if d(x) = 2 and d(y) = 1 then $G \cong G_3$. If d(x) = 2 and d(y) = 2, then G_4 or G_5. If d(x) = 3 and d(y) = 1, then $G \cong G_6$. If d(x) = 3 and d(y) = 2, then $G \cong G_7$ or G_8. In all the other cases, no new graph exists.

Subcase (b) Let <S> = $\overline{K}_2$. Since G is connected, there exists a vertex u_i in K_{n-2} which is adjacent to both the vertices x and y (or) u_i is adjacent to x and u_j for some $i \neq j$ is adjacent to y. In both the cases, {x, y, u_i, u_j} for $i \neq j$ is a γ_{fdd} -set n = 6. Hence K = K_4. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Without loss of generality, let u_1 be adjacent to both x and y. If d(x) = d(y) = 1 then $G \cong G_9$. In all the remaining cases, no new graph exits. Now without loss of generality, let u_1 be adjacent to x and u_2 be adjacent to y. If d(x) = d(y) = 1, then $G \cong G_{10}$. For all the remaining cases, no new graph exits.

Case (iv) Let $\gamma_{fdd} = n - 3$ and $\chi = n - 1$

Since $\chi = n - 1$, G contains a clique K on n – 1 vertices. Let x be adjacent to u_i for some i in K_{n-1}. Then $\{x, u_i, u_j\}$, for $i \neq j$ is a γ_{fdd}-set, so that n = 6. Hence $K = K_5$. Let u_1, u_2, u_3, u_4, u_5 be the vertices of K_5. Then x must be adjacent to only one vertices of K_5. Without loss of generality let x be adjacent to u_1. If d(x) = 1, then $G \cong G_{11}$.

Case (v) Let $\gamma_{fdd} = n - 4$ and $\chi = n$.

Since $\chi = n$ then $G = K_n$. But for K_n, $\gamma_{fdd} = 2$ so that n = 6. Hence $G \cong K_6$.

The authors are obtained a large classes of graphs with very lengthy proof for which $\gamma_{fdd}(G) + \chi(G) = 2n - 5$, $\gamma_{fdd}(G) + \chi(G) = 2n - 6$ and $\gamma_{fdd}(G) + \chi(G) = 2n - 7$.

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